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A COMPATIBLE TRIANGULAR PLATE BENDING FINITE ELEMENT

G. A. BUTLIN and R. FORD†

University of Leicester

NOTATION

a, b, c	lengths of sides of triangle
D	flexural rigidity
w	normal (out of plane) displacement
x, y	Cartesian in-plane co-ordinates
x', y'	oblique in-plane co-ordinates
X, Y	non-dimensional oblique co-ordinates
β, γ	angles of triangle (see Fig. 1)
ν	Poisson's ratio
Φ	2nd order Hermitian polynomial
Ψ	3rd order Hermitian polynomial

Superscripts

$A, B, C, D, E, F, G,$	refer to points of the triangle shown in Fig. 1.

Subscripts

r	tangential direction along CA (see Fig. 1)
s	normal direction to CA
t	tangential direction along BA
u	normal direction to BA
x	differentiation with respect to x
y	differentiation with respect to y

INTRODUCTION

THE development of a triangular finite element for plate flexure is a necessary step towards an element to analyse shell structures. Various attempts have been made at providing such an element. Some earlier ones [1–3] did not ensure compatibility between adjacent elements in their assumed displacement functions but more recently, compatible triangular elements have been developed [4–6]. There remained a need for a compatible triangular bending element which would yield results which would converge more rapidly to exact values.

Energy bounds, convergence and accuracy require individual consideration.

In order to ensure that the strain energy, associated with an approximate solution in which displacements are prescribed, is an overestimate of the exact strain energy, the assumed displacement fields must be compatible.

† Research Officer, Computing Branch, C.E.G.B.

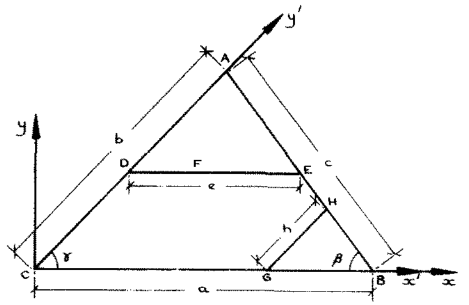


FIG. 1. Geometry of general triangle.

In order to allow eventual convergence to the exact solution as more elements are used, two conditions must be satisfied by the assumed displacement fields. These are (a) that uniform strain states should be allowed to occur and (b) that body movement should be allowed to occur without straining.

Accurate results can be achieved by careful consideration of the choice of the degrees of freedom for a particular structural element.

One of the authors demonstrated [7] the value of using high order polynomial displacement assumptions, where an acute angle is enclosed between two boundaries of a plate bending finite element.

As an example, a right-angled triangular compatible finite element was developed with extra parameters introduced along the hypotenuse. It was shown to produce solutions which converged more rapidly than other triangular elements, despite errors introduced by numerical integration in the calculation of the stiffness matrix. The present finite element is based on a similar displacement function as that used in [7], but developed for a general triangle. The stiffness matrix has been calculated in explicit form using programmed algebraic manipulation.

1. DERIVATION OF DISPLACEMENT FUNCTION

The effect, on the performance of a right-angled triangular bending element [7], of introducing fifth order polynomial variation of displacement along the hypotenuse suggested that quintic variation be used along all sides of a general triangle.

The choice of quintic variation of displacement along boundaries requires sufficient nodal parameters to define this polynomial. The choice here is:

$$w, \frac{\partial w}{\partial x}, \frac{\partial w}{\partial y}, \frac{\partial^2 w}{\partial x^2}, \frac{\partial^2 w}{\partial x \partial y}, \frac{\partial^2 w}{\partial y^2}$$

at each node. Hence:

$$\begin{aligned} w^D &= w^c\Psi_1(Y)+bw_r^c\Psi_2(Y)+b^2w_{rr}^c\Psi_3(Y)+w^A\Psi_4(Y)+bw_r^A\Psi_5(Y)+b^2w_{rr}^A\Psi_6(Y) \\ w^E &= w^B\Psi_1(Y)+cw_t^B\Psi_2(Y)+c^2w_{tt}^B\Psi_3(Y)+w^A\Psi_4(Y)+cw_t^A\Psi_5(Y)+c^2w_{tt}^A\Psi_6(Y) \end{aligned} \tag{1.1}$$

where w_{rr}^c is written for the second derivative w.r.t. r at point C etc. The variation of normal slope along a boundary is defined by a cubic polynomial in terms of the parameters at the nodes defining the boundary. Hence:

$$\begin{aligned} w_s^D &= w_s^C\Phi_1(Y)+bw_{rs}^C\Phi_2(Y)+w_s^A\Phi_3(Y)+bw_{rs}^A\Phi_4(Y) \\ w_u^E &= w_u^B\Phi_1(Y)+cw_{ut}^B\Phi_2(Y)+w_u^A\Phi_3(Y)+cw_{ut}^A\Phi_4(Y). \end{aligned} \tag{1.2}$$

The non-dimensional oblique Y co-ordinate is defined by

$$Y = y'/b = (y \operatorname{cosec} \gamma)/b \tag{1.3}$$

and the second, and third order Hermitian polynomials have been used. (See Fig. 2).

$$\begin{aligned} \Phi_1(r) &= (1-r)^2(1+2r) & \Psi_1(r) &= (1-r)^3(1+3r+6r^2) \\ \Phi_2(r) &= (1-r)^2 r & \Psi_2(r) &= (1-r)^3(r+3r^2) \\ \Phi_3(r) &= r^2(3-2r) & \Psi_3(r) &= \tfrac{1}{2}(1-r)^3 r^2 \\ \Phi_4(r) &= -r^2(1-r) & \Psi_4(r) &= r^3(10-15r+6r^2) \\ & & \Psi_5(r) &= -r^3(4-7r+3r^2) \\ & & \Psi_6(r) &= \tfrac{1}{2}r^3(1-2r+r^2). \end{aligned} \tag{1.4}$$

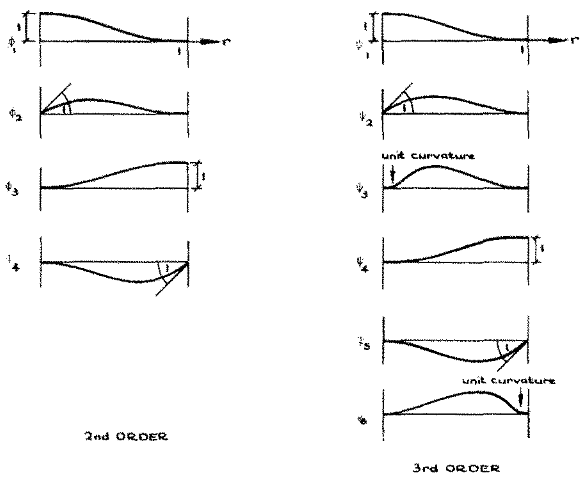


FIG. 2. Hermitian polynomials.

The displacement within the triangle is first defined by a third order polynomial in terms of the displacement and normal slope along the boundaries AB and AC.

$$w^F = w^D\Phi_1\left(\frac{x'}{e}\right)+ew_x^D\Phi_2\left(\frac{x'}{e}\right)+w^E\Phi_3\left(\frac{x'}{e}\right)+ew_x^E\Phi_4\left(\frac{x'}{e}\right) \tag{1.5}$$

where e defines the length DE

$$e = a(1-Y)$$

and the oblique co-ordinate x' is defined by

$$X = x'/a = (x-y \cot \gamma)/a. \tag{1.6}$$

Equation (1.5) requires the definition of w_x^D, and w_x^E. These quantities are known in terms of the normal and tangential slopes along the boundaries.

$$\begin{aligned} w_x^D &= w_s^D \sin\gamma + \frac{1}{b}\cdot\frac{\mathrm{d}w^D}{\mathrm{d}Y}\cos\gamma \\ w_x^E &= w_u^E \sin\beta - \frac{1}{c}\cdot\frac{\mathrm{d}w^E}{\mathrm{d}Y}\cos\beta. \end{aligned} \tag{1.7}$$

The function w^F has the required displacement and normal slope along AC and AB. However the displacement and normal slope along CB are not of the required form, and hence correction functions must be added to w^F to form the displacement function for the triangle.

Firstly, the displacement along CB is required to be

$$\begin{aligned} w^G = {} & w^c\Psi_1(X) + aw_x^c\Psi_2(X) + a^2 w_{xx}^c\Psi_3(X) + w^B\Psi_4(X) \\ & + aw_x^B\Psi_5(X) + a^2 w_{xx}^B\Psi_6(X). \end{aligned} \tag{1.8}$$

A correction function for displacement is described by a cubic polynomial along GH.

$$w_{c_1} = C_1(X)\Phi_1\left(\frac{y'}{h}\right) \tag{1.9}$$

where h is the length of GH

$$h = b(1-X) \tag{1.10}$$

and

$$C_1(X) = w^G - (w^F)_{Y=0}. \tag{1.11}$$

To correct the normal slope along CB a further cubic polynomial along GH is required. The normal slope should be described by

$$w_y^G = w_y^C\Phi_1(X) + aw_{xy}^C\Phi_2(X) + w_y^B\Phi_3(X) + aw_{xy}^B\Phi_4(X) \tag{1.12}$$

Hence, a correction function is

$$w_{c_2} = hC_2(X)\Phi_2\left(\frac{y'}{h}\right) \tag{1.13}$$

where

$$C_2(X) = w_y^G \sin\gamma + \frac{1}{a}\frac{\mathrm{d}w^G}{\mathrm{d}X}\cdot\cos\gamma - \frac{1}{b}\left(\frac{\mathrm{d}w^F}{\mathrm{d}Y}\right)_{Y=0}. \tag{1.14}$$

The final displacement function is obtained by addition of these corrections to w^F.

$$w = w^F + w_{c_1} + w_{c_2}. \tag{1.15}$$

This function now satisfies the original requirements. Consideration was given in the first instance to the nature of the displacement function and parameters. Inherent within this formulation is the unique definition of displacements along each boundary by only those displacement parameters at the ends of that boundary. Thus compatibility of displacements between elements is ensured by compatibility between nodal parameters. This formulation also ensures unique definition of displacements along lines within the element such that rigid body movements described along the boundaries produce internal displacement fields which do not strain the element. Moreover, uniform bending in each of two orthogonal directions and uniform twist can be represented by this function (illustrated in [7]).

2. CALCULATION OF THE STIFFNESS MATRIX

The calculation of the coefficients of the stiffness matrix is achieved by the standard procedure of minimisation of the energy integral. Hence,

$$K_{ij} = D\int\int_{\text{area}} \frac{\partial^2 w_i}{\partial x^2}\cdot\frac{\partial^2 w_j}{\partial x^2}+\frac{\partial^2 w_i}{\partial y^2}\cdot\frac{\partial^2 w_j}{\partial y^2}+\nu\frac{\partial^2 w_i}{\partial y^2}\cdot\frac{\partial^2 w_j \nu}{\partial x^2}+\nu\frac{\partial^2 w_i}{\partial x^2}\cdot\frac{\partial^2 w_j}{\partial y^2}$$
$$+2(1-\nu)\frac{\partial^2 w_i}{\partial x\partial y}\cdot\frac{\partial^2 w_j}{\partial x\partial y}\,\mathrm{d}x\,\mathrm{d}y. \tag{2.1}$$

Since oblique co-ordinates were used to find w, it is convenient to express (2.1) in terms of oblique co-ordinates. The transformation for derivatives is obtained from (1.3 and 1.6)

$$\frac{\partial}{\partial x} = \frac{1}{a}\frac{\partial}{\partial X}$$
$$\frac{\partial}{\partial y} = -\frac{\cot\gamma}{a}\frac{\partial}{\partial X}+\frac{\operatorname{cosec}\gamma}{b}\frac{\partial}{\partial Y}. \tag{2.2}$$

The Jacobian is given by

$$J = \begin{vmatrix} a & b\cos\gamma \\ 0 & b\sin\gamma \end{vmatrix} = ab\sin\gamma. \tag{2.3}$$

Using (2.2) and (2.3) in (2.1) it is found that

$$K_{ij} = D\,.\,ab\sin\gamma\int_{\text{area}}(C1\,.\,W1+C2\,.\,W2+C3\,.\,W3+C4\,.\,W4)\,\mathrm{d}X\,\mathrm{d}Y \tag{2.4}$$

where

$$W1 = \frac{1}{a^2b^2}\frac{\partial^2 w_i}{\partial X\partial Y}\cdot\frac{\partial^2 w_j}{\partial X\partial Y}$$
$$W2 = \frac{1}{a^4}\frac{\partial^2 w_i}{\partial X^2}\cdot\frac{\partial^2 w_j}{\partial X^2}+\frac{1}{b^4}\frac{\partial^2 w_j}{\partial Y^2}\cdot\frac{\partial^2 w_i}{\partial Y^2}$$
$$W3 = \frac{1}{a^2b^2}\left(\frac{\partial^2 w_i}{\partial X^2}\cdot\frac{\partial^2 w_i}{\partial Y^2}+\frac{\partial^2 w_j}{\partial Y^2}\cdot\frac{\partial^2 w_i}{\partial X^2}\right)$$
$$W4 = \frac{1}{a^3b}\frac{\partial^2 w_j}{\partial X^2}\cdot\frac{\partial^2 w_i}{\partial X\partial Y}+\frac{1}{ab^3}\frac{\partial^2 w_i}{\partial Y^2}\cdot\frac{\partial^2 w_j}{\partial X\partial Y}+\frac{1}{a^3b}\frac{\partial^2 w_j}{\partial Y\partial X}\cdot\frac{\partial^2 w_i}{\partial X^2}+\frac{1}{ab^3}\frac{\partial^2 w_j}{\partial Y^2}\cdot\frac{\partial^2 w_i}{\partial X\partial Y}$$

and,

$$
\begin{aligned}
C1 &= 2\,\mathrm{cosec}^2\,\gamma(2\,\mathrm{cosec}^2\,\gamma-(1+\nu))\\
C2 &= \mathrm{cosec}^4\,\gamma\\
C3 &= \mathrm{cosec}^2\,\gamma(\mathrm{cosec}^2\,\gamma-(1-\nu))\\
C4 &= -2\cot\gamma\,\mathrm{cosec}^3\,\gamma.
\end{aligned}
$$

Each of the integrals in (2.4) has been computed in general algebraic form in terms of the parameters of the triangle. Stiffness matrices for triangular elements of differing geometry are then evaluated by successive substitution for the parameters of the triangle into the

TABLE 1. LOAD VECTOR DUE TO UNIFORM LOAD ON ELEMENT

$$F_1 = 0{\cdot}1428571\,ab\sin\gamma+0{\cdot}02380952\,a^2(\sin\gamma\cos\gamma+\sin\beta\cos\beta)$$

$$F_2 = 0{\cdot}01904762\,a(b^2\sin\gamma\cos\gamma-c^2\sin\beta\cos\beta)+0{\cdot}006349206\,a^2(b\cos^2\gamma\sin\gamma-c\cos^2\beta\sin\beta) +0{\cdot}005555556\,a^2(c\sin^3\beta-b\sin^3\gamma)$$

$$F_3 = -0{\cdot}03809524\,ab^2\sin^2\gamma-0{\cdot}01190476\,a^2(b\cos\gamma\sin^2\gamma+c\cos\beta\sin^2\beta)$$

$$F_4 = 0{\cdot}001785714\,a(b^3\cos^2\gamma\sin\gamma+c^3\cos^2\beta\sin\beta)+0{\cdot}0005952381\,a^2(b^2\cos^3\gamma\sin\gamma+c^2\cos^3\beta\sin\beta) -0{\cdot}001388889\,a^2(b^2\cos\gamma\sin^3\gamma+c^2\cos\beta\sin^3\beta)$$

$$F_5 = 0{\cdot}003571429\,a(b^3\cos\gamma\sin^2\gamma-c^3\cos\beta\sin^2\beta)+0{\cdot}002579365\,a^2(b^2\sin^2\gamma\cos^2\gamma-c^2\sin^2\beta\cos^2\beta) +0{\cdot}001388889\,a^2(c^2\sin^4\beta-b^2\sin^4\gamma))$$

$$F_6 = 0{\cdot}003571429\,ab^3\sin^3\gamma+0{\cdot}001984127\,a^2(b^2\cos\gamma\sin^3\gamma+c^2\cos\beta\sin^3\beta)$$

$$F_7 = 0{\cdot}1666667\,ab\sin\gamma+0{\cdot}02380952\,(b^2\sin\gamma\cos\gamma-a^2\sin\beta\cos\beta)$$

$$F_8 = 0{\cdot}02539683\,ac^2\sin\beta\cos\beta+0{\cdot}006349206\,a(b^2\sin\gamma\cos\gamma-ac\cos^2\beta\sin\beta)+0{\cdot}005555556\,a^2c\sin^3\beta +0{\cdot}01904762\,a^2b\sin\gamma$$

$$F_9 = 0{\cdot}03095238\,ab^2\sin^2\gamma-0{\cdot}01190476\,a^2c\cos\beta\sin^2\beta$$

$$F_{10} = 0{\cdot}002380952\,ac^3\cos^2\beta\sin\beta+0{\cdot}0005952381\,a^2(b^2\sin\gamma\cos\gamma-c^2\cos^3\beta\sin\beta) +0{\cdot}001388889\,a^2c^3\cos\beta\sin^3\beta+0{\cdot}001785714\,a^3b\sin\gamma$$

$$F_{11} = -0{\cdot}004761905\,ac^3\sin^2\beta\cos\beta+0{\cdot}002579365\,a^2c^2\cos^2\beta\sin^2\beta -0{\cdot}001388889\,a^2(b^2\sin^2\gamma+c^2\sin^4\beta)$$

$$F_{12} = 0{\cdot}002380952\,ab^3\sin^3\gamma-0{\cdot}001984127\,a^2c^2\cos\beta\sin^3\beta$$

$$F_{13} = 0{\cdot}1904762\,ab\sin\gamma-0{\cdot}02380952\,(a^2\sin\gamma\cos\gamma+b^2\sin\gamma\cos\gamma)$$

$$F_{14} = -0{\cdot}01904762\,ab^2\sin\gamma\cos\gamma+0{\cdot}01190476\,a^2b\cos^2\gamma\sin\gamma-0{\cdot}03095238\,a^2b\sin\gamma$$

$$F_{15} = 0{\cdot}03095238\,ab^2\sin^2\gamma-0{\cdot}01190476\,a^2b\cos\gamma\sin^2\gamma$$

$$F_{16} = 0{\cdot}002380952\,a(b^3\cos^2\gamma\sin\gamma+a^2b\sin\gamma)-0{\cdot}001984127\,a^2b^2\cos^3\gamma\sin\gamma+0{\cdot}0007936508\,a^2b^2\cos\gamma\sin\gamma$$

$$F_{17} = 0{\cdot}004761905\,ab^2\cos\gamma\sin^3\gamma-0{\cdot}003968254\,a^2b^2\sin^2\gamma\cos^2\gamma+0{\cdot}002777778\,a^2b^2\sin^2\gamma$$

$$F_{18} = 0{\cdot}002380952\,ab^3\sin^3\gamma-0{\cdot}001984127\,a^2b^2\cos\gamma\sin^3\gamma$$

where

$$
\begin{bmatrix} F_1\\ F_2\\ \vdots\\ F_{18}\end{bmatrix} = \begin{bmatrix} F_A\\ F_B\\ F_C\end{bmatrix} \quad\text{and}\quad F_A = \begin{bmatrix} F_w\\ F_{\theta y}\\ F_{\theta x}\\ F_{wxx}\\ F_{wxy}\\ F_{wyy}\end{bmatrix} A.
$$

algebraic expressions for the stiffness coefficients. Since the integration has been carried out in algebraic form, the only error is due to rounding. Numerical procedures, such as Gauss Quadrature, have been found to converge very slowly, when used to integrate the expression (2.1), or in fact any function which involves negative powers. The stiffness matrix in [7] was evaluated using Gaussian quadrature, with basically the same displacement shapes used here. The results shown in Fig. 5 are thought to reflect the benefit of accurate integration.

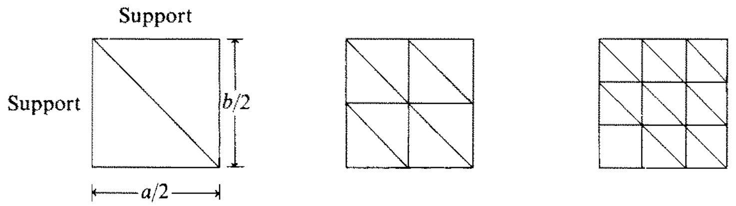
On the IBM 360/75 a time of approximately 0·15 of a second is required to evaluate the coefficients of the stiffness matrix of the triangular element, using the previously calculated algebraic formulae (which require 16 K words of core storage).

3. CALCULATION OF LOAD VECTOR

This programme for algebraic manipulation was also used to calculate the load vector for the triangular element, under a uniform load, by exact integration of the displacement function w for each nodal parameter. The results of this computation are shown in Table 1.

TABLE 2. RESULTS FOR CENTRAL DEFLECTION OF PLATES

Element subdivision (Symmetrical w.r.t. top left corner)



A *B* *C*

Case 1. Square plate, four sides simply supported, central point load.
Case 2. Square plate, four sides clamped, central point load.
Case 3. Square plate, four sides simply supported, uniform load.
Case 4. Square plate, four sides clamped, uniform load.
Case 5. Rectangular plate $b/a = 2$, four sides simply supported, central point load.
Case 6. Rectangular plate $b/a = 2$, four sides clamped, central point load.

The results show the central deflection as a deflection coefficient α, where $w = a^2\alpha P/D$ for point load (P) and $w = a^4\alpha q/D$ for uniform load (of intensity q).

Degree of element subdivision	*A*	*B*	*C*	Timoshenko's value [8]
Case 1	0·011490	0·011574	0·011590	0·01160
Case 2	0·005535	0·005582	0·005600	0·00560
Case 3	0·004061	0·004062	0·004063	0·00406
Case 4	0·001261	0·001264	0·001265	0·00126
Case 5	0·015978	0·016416	0·016476	0·01651
Case 6	0·006871	0·007125	0·007183	0·00722

4. DISCUSSION OF RESULTS

Results for a square plate under uniform and concentrated load, and a rectangular plate under concentrated load, are given in Table 2 and Figs. 3 and 4 for simply-supported and clamped edge conditions. It can be seen that the present triangular element yields a consistently accurate approximation, even when only 2 elements are used. The solutions are also associated with an upper bound on the strain energy and converge rapidly in all cases tested.

Comparison, where possible, has been made with other compatible triangular bending elements, [4–6].

This formulation of the displacement function is non-symmetric since it requires the choice of oblique co-ordinates along two sides of the triangle. To achieve stiffness coefficients in general form which would be independent of the choice of axes one could derive three sets of coefficients for the three possible choices of axes and then sum them and divide them by three with the programme for algebraic manipulation. However it is easier to

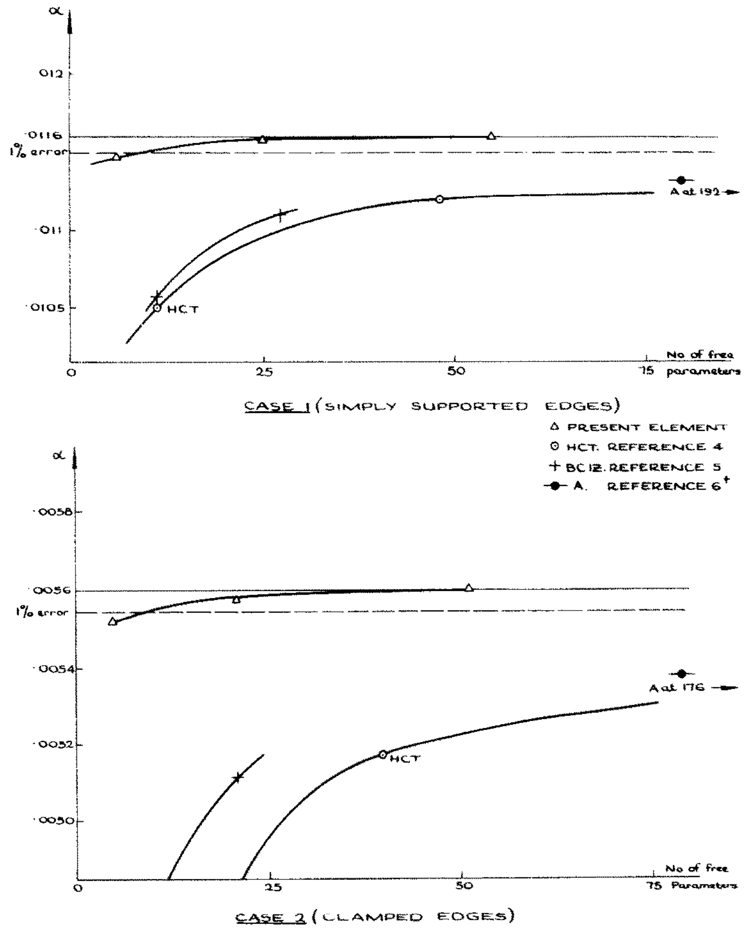


FIG. 3. Central deflection of square plate under point load.

† The number of free parameters was not quoted in this paper; the figures quoted are the author's interpretation.

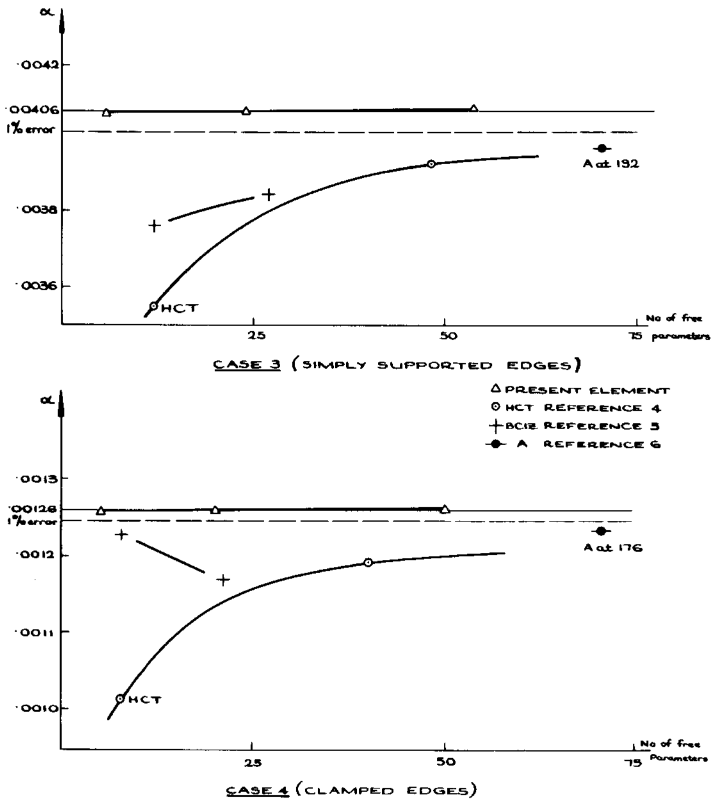


Fig. 4. Central deflection of square plate under uniform load.

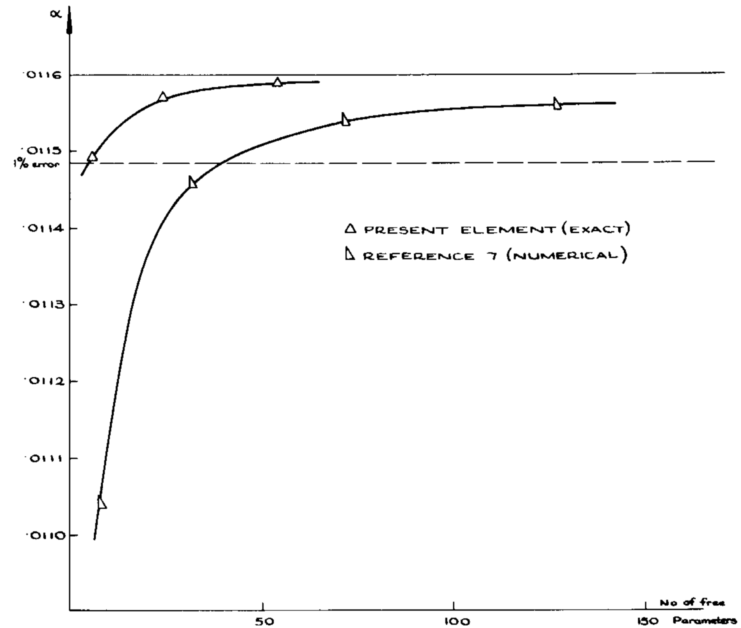


Fig. 5. Comparison of central deflection of simply supported square plate under point load with numerical and exact integration.

substitute three sets of numerical values into the general coefficients and average the results. When this was tried for an arbitrary triangle the stiffness coefficients corresponding to the three choices of axes produced differences in the coefficients of the order of 0·004 per cent on average and 0·1 per cent at most. The increase in computing time to average the coefficients is of course three fold.

5. CONCLUSIONS

The element presented in this paper appears to provide increased accuracy in the solution of plate flexure problems, and serves to emphasise the value of careful choice of boundary displacement variation assumptions.

The computer programmed algebra not only provides exact integration, but also enables the rapid evaluation of stiffness matrices for triangular elements of differing geometry.

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